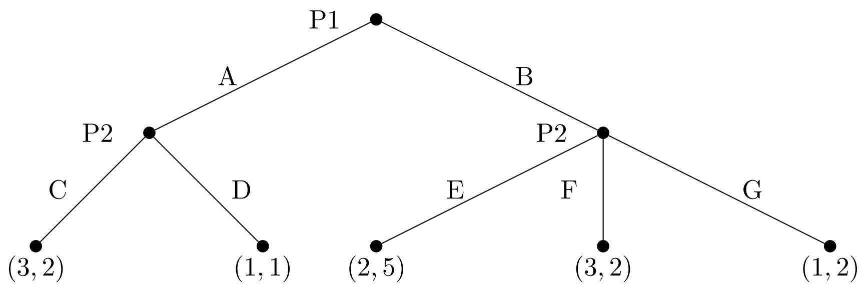
Microeconomics III
Autumn 2020

Exam Solutions

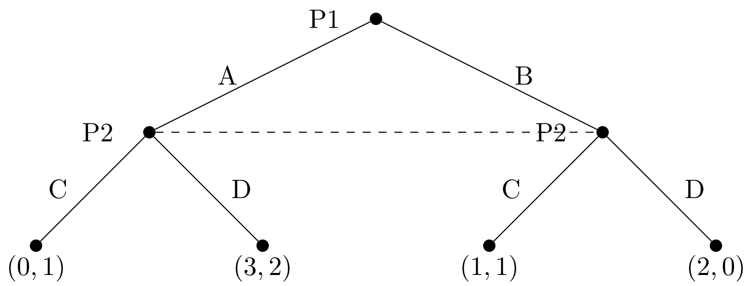
1. Find all pure strategy subgame perfect Nash equilibria of the following games (the first payoff is that of player 1, the second payoff that of player 2). Remember to define full equilibrium strategies.

   (a) (3 points)



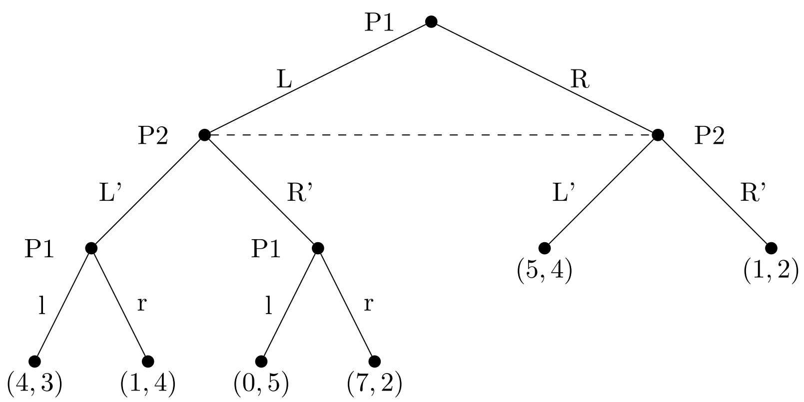
   Solution. SPNE= (A, CE).

   (b) (3 points)



   Solution. SPNE= $\{(A, D); (B, C)\}$. (Notice that the game is essentially static; there is also a mixed NE $((0.5, 0.5), (0.5), (0.5))$.)

   (c) (4 points)



   Solution. SPNE= (Rlr, L').

2. Two firms choose simultaneously their levels of production, q_1 and q_2. Firm 2 already has produced $\bar{q} \leq 1/2$ and will have to sell that quantity no matter what. q_2 is what Firm 2 can produce additionally so that its total supply is $\bar{q} + q_2$. Products are identical and the inverse demand is $P(q_1, q_2, \bar{q}) = 1 - q_1 - q_2 - \bar{q}$. The marginal cost of production is 0 for both firms.

(a) (1 point) Write the game in normal form.

Solution. $(\{Firm1, Firm2\}, \mathbf{R}_+ \times \mathbf{R}_+, (u_1(q_1, q_2) = P(q_1, q_2, \bar{q})q_1, u_2(q_1, q_2) = P(q_1, q_2, \bar{q})(q_2 + \bar{q}))$ where

- $\{Firm1, Firm2\}$ is the set of players,
- $\mathbf{R}_+$ is the set of pure strategies/actions,
- u_i is the payoff function for Firm i.

(b) (4 points) For each $\bar{q} \in [0, 1/2]$, find the Nash equilibrium of the game. Does the storage help or hurt Firm 2 (write 1-2 sentences)?

Solution. Let's start with Firm 1:

$$u_1(q_1, q_2; \bar{q}) = (1 - q_1 - q_2 - \bar{q})q_1.$$

The first order condition:

$$1 - q_2 - \bar{q} - 2q_1 = 0 \iff B_1(q_2; \bar{q}) = \max\{\frac{1 - q_2 - \bar{q}}{2}, 0\}.$$

Similarly for Firm 2:

$$u_2(q_1, q_2; \bar{q}) = (1 - q_1 - q_2 - \bar{q})(q_2 + \bar{q}).$$

The first order condition:

$$1 - q_1 - 2q_2 - 2\bar{q} = 0 \iff B_2(q_1; \bar{q}) = \max\{\frac{1 - q_1 - 2\bar{q}}{2}, 0\}.$$

Plugging in $B_1(q_2; \bar{q})$ to the first order condition of Firm 2 yields:

$$1 - \frac{1 - q_2 - \bar{q}}{2} - 2q_2 - 2\bar{q} = 0 \iff q_2 = \frac{1}{3} - \bar{q}.$$

Solution: the Nash equilibrium is $q_2^* = \max\{\frac{1}{3} - \bar{q}, 0\}$ and $q_1^* = \frac{1 - \max\{1/3, \bar{q}\}}{2} = \min\{\frac{1}{3}, \frac{1-\bar{q}}{2}\}$.

If $\bar{q} \leq 1/3$, it does not affect total amount of sales: Firm 1 sells and produces 1/3 and Firm 2 sells 1/3 and hence produces $q_2 = 1/3 - \bar{q}$. In this case, the storage does not affect either firm's profits.

If $\bar{q} > 1/3$, it affects total sales and the division of sales between the two firms: Firm 1 sells and produces $(1 - \bar{q})/2$ and Firm 2 does not produce more, $q_2 = 0$, and sells $\bar{q}$. In this case, the storage makes Firm 2 better off and Firm 1 worse off. To see the former, calculate the payoff for Firm 2 as a function of $\bar{q}$:

$$u_2^*(\bar{q}) = u_2(q_1^*, q_2^*; \bar{q}) = (1 - (1 - \bar{q})/2 - \bar{q})\bar{q} = 1/2(1 - \bar{q})\bar{q}$$

$u_2^{*\prime}(\bar{q}) = 1/2(1 - 2\bar{q}) > 0$ for all $\bar{q} < 1/2$.
Large storage helps Firm 2 because it makes Firm 1 to produce less. In fact, the storage enables that Firm 2 can act like the Stackelberg leader: $q = 1/2$ would be the quantity chosen by the first mover in the Stackelberg game.

(c) (1 point) Assume now that only Firm 2 knows $\bar{q}$ and that it is 0 with probability 0.5 and 1/2 with probability 0.5. Write the game of incomplete information in normal form.

Solution. $(\{Firm1, Firm2\}, \mathbf{R}_+ \times \mathbf{R}_+, (\{t_1\}, \{0, 1/2\}), Pr(\bar{q} = 1/2) = 0.5,$
$(u_1(q_1, q_2, \bar{q}) = P(q_1, q_2, \bar{q})q_1, u_2(q_1, q_2, \bar{q}) = P(q_1, q_2, \bar{q})(q_2 + \bar{q}))$ where

- $\{Firm1, Firm2\}$ is the set of players,
- $\mathbf{R}_+$ is the set of pure strategies/actions,
- $(\{t_1\}, \{0, 1/2\})$ is the set of types (trivial for Firm 1),
- $Pr(\bar{q} = 1/2) = 0.5$ pins down the type distribution,
- $u_i(\cdot, \bar{q})$ is the payoff function for Firm i when Firm 2's type is $\bar{q}$.

(d) (4 points) Find the Bayes Nash equilibrium of the game with incomplete information. Does incomplete information help or hurt Firm 2 compared to the case where also Firm 1 observes $\bar{q}$ (write 1-2 sentences)?

Solution. Let's start with Firm 2. We can use the best-response function from part (b): $B_2(q_1; \bar{q}) = \max\{\frac{1-q_1-2\bar{q}}{2}, 0\}$. If $\bar{q} = 0$, Firm 2 has total quantity $(1 - q_1)/2$ and if $\bar{q} = 1/2$, Firm 2 has total quantity 1/2. (For the latter, observe that $(1 - q_1 - 2(1/2))/2 \leq 0$ for all $q_1 \geq 0$.)

Firm 1 then believes that $q_2 + \bar{q} = (1 - q_1)/2 =: q_2^L$ with probability 0.5 and $q_2 + \bar{q} = 1/2 =: q_2^H$ with probability 0.5. Firm 1's expected payoff is hence:

$$\mathbb{E}[u_1(q_1, q_2(\bar{q}), \bar{q})] = 0.5(1 - q_1 - q_2^L)q_1 + 0.5(1 - q_1 - q_2^H)q_1.$$

The first order condition:

$$1 - 0.5(q_2^H + q_2^L) - 2q_1 = 0 \iff 1 - 0.5(1/2 + (1 - q_1)/2) - 2q_1 = 0 \iff q_1* = 2/7.$$

We can then plug q_1^* back to $B_2(q_1; \bar{q})$ to get $q_2^L = q_2^*(0) = 5/14$.

Solution: The Bayes Nash equilibrium is $q_1^* = 2/7$, $q_2^*(0) = 5/14$, $q_2^*(1/2) = 0$.

Firm 2 makes expected profits equal to:

$$\mathbf{E}[u_2(q_1^*, q_2^*(\bar{q}), \bar{q})] = 0.5(1 - 2/7 - 5/14)5/14 + 0.5(1 - 2/7 - 1/2)1/2 = 23/196.$$

We can use the result from part (b) to get the complete information benchmark. If the storage is 0 w.p. 0.5 and 1/2 w.p. 1/2 but Firm 1 observes the realization, Firm 2 makes expected profits equal to:

$$0.5u_2^*(0) + 0.5u_2^*(1/2) = 0.5(1 - 1/3 - 1/3)1/3 + 0.5(1/2(1 - 1/2)1/2) = 17/144 > 23/196.$$

Hence, Firm 2 is better off if also Firm 1 observes $\bar{q}$. Incomplete information hurts Firm 2 because, compared to the complete information benchmark, Firm 1 produces less when Firm 2 sells less (when $\bar{q} = 0$) and more when Firm 2 sells more (when $\bar{q} = 1/2$).

3. Consider the following (simultaneous move/ static) stage game:

		Player 2		
		a	b	c
Player 1	A	1, 6	3, 2	6, 1
	B	0, 2	4, 4	5, 2
	C	2, 3	5, 1	2, 0

(a) (4 points) Find all (pure and mixed) Nash equilibria of the stage game. Make sure that you have argued that there are no other Nash equilibria. Remember to argue that there cannot be any other Nash equilibria.

Solution. Let's follow the cookbook from the last lecture.

1) Best responses:

		Player 2		
		a	b	c
Player 1	A	$1,\underline{6}$	$3,2$	$\underline{6},1$
	B	$0,2$	$4,\underline{4}$	$5,2$
	C	$\underline{2},\underline{3}$	$\underline{5},1$	$2,0$

We have found one pure strategy NE: (C, a).

2) Can anything be dominated: We know that actions B and c are not best responses to any pure actions and hence they may be strictly dominated.
3) IESDS: Both actions a and b strictly dominate c. After eliminating c, C strictly dominates A and B. Then, a strictly dominates b. (C, a) is the unique pure actions profile survives IESDS and hence it is the unique Nash equilibrium.

Solution: NE$= (C, a)$. There cannot be other Nash equilibria because no other strategies survive IESDS.

(b) (6 points) Now, consider infinitely many times repeated game with discount factor $\delta < 1$. Show that if δ is close to 1, there exists a subgame perfect Nash equilibrium where (B, b) is played in every period on the equilibrium path. Construct such an equilibrium, i.e. find equilibrium strategies.

Solution. We can use a trigger strategy that uses the stage Nash equilibrium as a punishment. The suggested equilibrium strategies are:

$$\sigma_1^t = \begin{cases} B & \text{if always } (B,b) \text{ or } t = 1 \\ C & \text{otherwise} \end{cases}$$

$$\sigma_2^t = \begin{cases} b & \text{if always } (B,b) \text{ or } t = 1 \\ a & \text{otherwise} \end{cases}$$

Let's check that the suggested strategy profile is a subgame perfect Nash equilibrium. Thanks to the one-shot deviation principle, it is enough to check that neither player has a profitable one-shot deviation.

The continuation value on the equilibrium path (after always (B, b)) is $4\delta/(1-\delta)$ for both players. Off the equilibrium path (after any player has deviated) it is $2\delta/(1-\delta)$ for Player 1 and $3\delta/(1-\delta)$ for Player 2. Because the continuation payoff is larger on path, deviation can be profitable only if it gives a higher payoff in the period when the deviation happens.

The best one-shot deviation for Player 1 on path is to play C. Action b is the static best response to B and hence Player 2 clearly does not want to deviate. Similarly, neither player wants to deviate off path (after a deviation has occurred) because they play stage NE and hence best respond to each other's actions. Player 1 does not want to deviate on path if and only if

$$4 + 4\delta/(1-\delta) \geq 5 + 2\delta/(1-\delta) \iff \delta \geq 1/3.$$

Solution: (σ_1^t, σ_2^t) is a SPNE whenever δ is large enough (namely $\geq 1/3$).

EXTRA (+2 points) Solve the following normal form game by using the iterative elimination of strictly dominated strategies - remember also to eliminate strategies that are dominated by a mixed strategy. Explain briefly each step (1 sentence).

		Player 2		
		a	b	c
Player 1	A	$1,6$	$3,2$	$1,5$
	B	$1,2$	$2,4$	$3,3$
	C	$2,3$	$5,1$	$0,0$

Solution.

Actions A and c are not best responses to any pure actions and hence they may be strictly dominated. It turns out that c cannot be eliminated even if we consider mixed actions.

Round 1: A is strictly dominated by a mixed action $(0, 0.5, 0.5)$:

$$1 < 0.5 \cdot 1 + 0.5 \cdot 2 = 1.5$$
$$3 < 0.5 \cdot 2 + 0.5 \cdot 5 = 3.5$$
$$1 < 0.5 \cdot 3 + 0.5 \cdot 0 = 1.5$$

(Alternatively, you may use abstract p to denote the probability that the mixed actions assigns on B and solve for the range of p that satisfy $1 < p1+(1-p)2 = 1.5$, $3 < p2+(1-p)5$, and $1 < p3+(1-p)0$.)

Round 2: After eliminating A, b strictly dominates c.

Round 3: Then, C strictly dominates B.

Round 4: After that a strictly dominates b.

Solution: (C, a) is the unique actions profile that survives IESDS. (Hence, it is also the unique Nash equilibrium.)